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A VARIATIONAL ANALYSIS OF SMALL FINITE DEFORMATIONS OF PRETWISTED ELASTIC BEAMS†

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Abstract—The principle of minimum potential energy, in conjunction with a suitably assumed strain energy density and in conjunction with suitably assumed expressions for strain and displacement components, is used to derive a system of non-linear ordinary differential equations for the problem of stretching, shearing, bending, twisting and warping of *pre*-twisted originally straight beams. Two problems which are considered in some detail concern the combined finite stretching and twisting of beams with doubly symmetric cross-section, and the influence of pretwist on shear and twist center locations.

INTRODUCTION

The analysis which follows attempts a synthesis of results for stretching, bending, twisting, and warping of prismatical beams as obtained in [3, 5], and of results by Krenk [2] and Hodges[1] for the linear and nonlinear theory of pretwisted beams.

While there are similarities in our approach and in the approaches in [1, 2], insofar as use of the principle of minimum potential energy is concerned, and insofar as use of the St. Venant torsional warping function for the introduction of the warping stiffness effect into the ensuing one-dimensional theory is concerned, there are also differences, as will be apparent from a comparison of the respective publications.

The present study limits itself to the discussion of two specific examples of application. The first of these is the problem of finite stretching and twisting of beams with doubly symmetric cross-section, including consideration of the effect of end section warping restraint. The second is the problem of cantilever torsion and flexure within the range of applicability of linear theory with a view towards establishing the influence of pretwist on twist and shear center locations.

ENERGY FUNCTIONALS AND DISPLACEMENT MODES

We begin as in [3] with the stipulation that an adequate three-dimensional strain energy expression for a beam with originally straight z-axis and cartesian cross sectional coordinates x, y is of the form

$$\Pi_s = \iiint U(\bar{\epsilon}_z, \bar{\gamma}_{xz}, \bar{\gamma}_{yz})\, dx\, dy\, dz. \tag{1}$$

In this, we assume that the restriction to problems of small finite deformations for sufficiently slender beams implies the appropriateness of the use of the abbreviated Green strain formulas

$$\bar{\epsilon}_z = \bar{w}_{,z} + \tfrac{1}{2}\bar{u}_{,z}^2 + \tfrac{1}{2}\bar{v}_{,z}^2, \tag{2a}$$

$$\bar{\gamma}_{xz} = \bar{u}_{,z} + \bar{w}_{,x} + \bar{u}_{,z}\,\bar{u}_{,x} + \bar{v}_{,z}\,\bar{v}_{,x}, \qquad \bar{\gamma}_{yz} = \bar{v}_{,z} + \bar{w}_{,y} + \bar{v}_{,z}\bar{v}_{,y} + \bar{u}_{,z}\bar{u}_{,y}. \tag{2b}$$

In order to be able to use the variational equation for displacements which is associated with eqns (1) and (2) for the derivation of an approximate one-dimensional beam theory we assume as before as approximations for the cartesian displacement

† Dedicated to the memory of Alicia Golebiewska-Herrmann, who will long be remembered by those who had the privilege of knowing her.

components $\bar{u}$, $\bar{v}$, $\bar{w}$

$$\bar{u} = u - y\theta, \qquad \bar{v} = v + x\theta, \qquad \bar{w} = w + x\alpha + y\beta + \lambda\phi, \tag{3}$$

with u, v, θ, w, α, β, λ being seven functions of z only, and with ϕ being a suitably assumed function of x, y, and z.

In our earlier analysis of prismatical beams, ϕ was taken to be the St. Venant warping function for torsion, with differential equation $[G(\phi_{,x} - y)]_{,x} + [G(\phi_{,y} + x)]_{,y} = 0$ and boundary condition $(\phi_{,x} - y)\,dx - (\phi_{,y} + x)\,dy = 0$. Here we modify our definition of ϕ by first introducing rotated cross-sectional coordinates ξ, η, involving an angle of pretwist $\omega(z)$, through the relations

$$x = \xi \cos\omega - \eta \sin\omega, \qquad y = \eta\cos\omega + \xi\sin\omega, \tag{4}$$

and by then stipulating that ϕ be the warping function for the rotated cross-section, with differential equation

$$[G(\phi_{,\xi} - \eta)]_{,\xi} + [G(\phi_{,\eta} + \xi)]_{,\eta} = 0, \tag{5a}$$

and boundary condition

$$f(\xi, \eta) = 0; \qquad (\phi_{,\xi} - \eta)\,d\xi - (\phi_{,\eta} + \xi)\,d\eta = 0, \tag{5b}$$

with $G = G(\xi, \eta)$ a given non-negative function and with (5a,b) implying the integral relations

$$\iint (\eta - \phi_{,\xi})G\,d\xi\,d\eta = \iint (\xi + \phi_{,\eta})G\,d\xi\,d\eta = 0 \tag{5c}$$

and

$$\iint (\eta\phi_{,\xi} - \xi\phi_{,\eta})G\,d\xi\,d\eta = \iint (\phi_{,\xi}^2 + \phi_{,\eta}^2)G\,d\xi\,d\eta. \tag{5d}$$

We note, for subsequent use, that the determination of ϕ, as in (5a,b), allows us to set, without loss of generality,

$$\iint \phi E\,d\xi\,d\eta = 0, \tag{5e}$$

with a Young modulus function $E = E(\xi, \eta)$.

Having the above definition of ϕ, it follows that the effect of pretwist will manifest itself in the one-dimensional theory which is to be established by way of modifying the approximate normal strain expression $\bar{\epsilon}_z$ of prismatical beam analysis, through the appearance of one additional term, involving a factor $\phi_{,z}$, as follows:

$$\bar{\epsilon}_z = w' + \tfrac{1}{2}(u')^2 + \tfrac{1}{2}(v')^2 + x(\alpha' + v'\theta') + y(\beta' - u'\theta') + \tfrac{1}{2}(x^2 + y^2)(\theta')^2 + \lambda'\phi + \lambda\phi_{,z}. \tag{6a}$$

At the same time, we have as before [5], except for negligible terms $x\theta\theta'$ and $y\theta\theta'$,

$$\bar{\gamma}_{xz} = \alpha + u' + \theta v' - y\theta' + \phi_{,x}\lambda, \tag{6b}$$

$$\bar{\gamma}_{yz} = \beta + v' - \theta u' + x\theta' + \phi_{,y}\lambda. \tag{6c}$$

In writing (6a,b,c) we take account of eqn (4) by observing that

$$\phi_{,x} = \phi_{,\xi}\cos\omega - \phi_{,\eta}\sin\omega, \qquad \phi_{,y} = \phi_{,\eta}\cos\omega + \phi_{,\xi}\sin\omega, \tag{7}$$

and

$$\phi_{,z} = (\phi_{,\xi}\eta - \phi_{,\eta}\xi)\omega', \tag{8}$$

with an associated area element change of $dx\,dy$ into $d\xi\,d\eta$ in the defining relation (1).

In order to proceed further, we now stipulate as expression for the potential energy of external distributed or concentrated loads, consistent with the displacement approximations in eqn (1),

$$\Pi_l = -\int (fw + q_x u + q_y v + t\theta + m_x\alpha + m_y\beta + r\lambda)\,dz - \Sigma(F_{zi}w + F_{xi}u + F_{yi}v + M_{xi}\alpha + M_{yi}\beta + T_i\theta + R_i\lambda)_{z=z_i}, \tag{9}$$

and we assume, for the sake of definiteness, a specific strain energy density function U of the form

$$U = \tfrac{1}{2}(E\bar{\epsilon}_z^2 + G\bar{\gamma}_{xz}^2 + G\bar{\gamma}_{yz}^2). \tag{10}$$

With eqns (1) to (10), the general results which are to be obtained now follow as a consequence of the variational equation

$$\delta(\Pi_s + \Pi_l) = 0, \tag{11}$$

with arbitrary δw, δu, δv, $\delta\alpha$, $\delta\beta$, $\delta\theta$, and $\delta\lambda$ for all values of z, excepting the effect of prescribed displacement boundary conditions.

DERIVATION OF ONE-DIMENSIONAL DIFFERENTIAL EQUATIONS AND BOUNDARY CONDITIONS

We write on the basis of eqns (1) and (2)

$$\delta\Pi_s = \iiint (\bar{\sigma}_z\delta\bar{\epsilon}_z + \bar{\tau}_{xz}\delta\bar{\gamma}_{xz} + \bar{\tau}_{yz}\delta\bar{\gamma}_{yz})\,d\xi\,d\eta\,dz, \tag{12}$$

and, on the basis of eqns (6a,b,c),

$$\delta\bar{\epsilon}_z = \delta\epsilon_F + x\delta\kappa_x + y\delta\kappa_y + (x^2 + y^2)\delta\epsilon_N + \phi\delta\kappa_R + (\eta\phi_{,\xi} - \xi\phi_{,\eta})\delta\gamma_H, \tag{13a}$$

$$\delta\bar{\gamma}_{xz} = \delta\gamma_x - y\delta\kappa_T + \phi_{,x}\delta\gamma_S, \qquad \delta\bar{\gamma}_{yz} = \delta\gamma_y + x\delta\kappa_T + \phi_{,y}\delta\gamma_S, \tag{13b}$$

where

$$\epsilon_F = w' + \tfrac{1}{2}(u')^2 + \tfrac{1}{2}(v')^2, \quad \kappa_x = \alpha' + v'\theta', \quad \kappa_y = \beta' - u'\theta', \tag{14a}$$

$$\epsilon_N = \tfrac{1}{2}(\theta')^2, \quad \kappa_R = \lambda', \quad \gamma_H = \omega'\lambda, \quad \kappa_T = \theta', \tag{14b}$$

$$\gamma_x = \alpha + u' + \theta v', \quad \gamma_y = \beta + v' - \theta u', \quad \gamma_S = \lambda. \tag{14c}$$

An introduction of eqns (13a,b), in conjunction with appropriate defining relations for one-dimensional cross-sectional stress measures, into eqn (12) reduces this relation to the one-dimensional form

$$\delta\Pi_s = \int (F\delta\epsilon_F + M_x\delta\kappa_x + M_y\delta\kappa_y + R\delta\kappa_R + H\delta\gamma_H + N\delta\epsilon_N + Q_x\delta\gamma_x + Q_y\delta\gamma_y + T\delta\kappa_T + S\delta\gamma_S)\,dz. \tag{15}$$

Equation (15) in conjunction with eqns (9), (11) and (14a,b,c) leads to a system of seven one-dimensional differential equations of equilibrium

$$F' + f = 0, \quad (T + v'M_x - u'M_y + \theta'N)' - v'Q_x + u'Q_y + t = 0, \tag{16}$$

$$(Q_x - \theta Q_y + u'F - \theta'M_y)' + q_x = 0, \quad M_x' - Q_x + m_x = 0, \tag{17}$$

$$(Q_y + \theta Q_x + v'F + \theta'M_x)' + q_y = 0, \quad M_y' - Q_y + m_y = 0, \tag{18}$$

$$R' - S - \omega'H + r = 0, \tag{19}$$

and to a system of one-dimensional stress boundary conditions

$$F = F_{zb}, \quad T + v'M_x - u'M_y + \theta'N = M_{zb}, \tag{20}$$

$$Q_x - \theta Q_y + u'F - \theta'M_y = F_{xb}, \quad M_x = M_{xb}, \tag{21}$$

$$Q_y + \theta Q_x + v'F + \theta'M_x = F_{yb}, \quad M_y = M_{yb}, \quad R = R_b, \tag{22}$$

or alternately, displacement boundary conditions

$$w = w_b, \quad \theta = \theta_b, \quad u = u_b, \quad \alpha = \alpha_b, \quad v = v_b, \quad \beta = \beta_b, \quad \lambda = \lambda_b. \tag{23}$$

Given the form (12) and (13), in conjunction with eqn (15), we have as defining relations for the one-dimensional stress measures in eqns (16) to (22),

$$(F, M_x, M_y, R, N, H) = \iint (1, x, y, \phi, x^2 + y^2, \eta\phi_{,\xi} - \xi\phi_{,\eta})\bar{\sigma}_z \, d\xi \, d\eta, \tag{24}$$

$$(Q_x, Q_y, T, S) = \iint (\bar{\tau}_{xz}, \bar{\tau}_{yz}, \bar{\tau}_{yz}x - \bar{\tau}_{xz}y, \bar{\tau}_{xz}\phi_{,x} + \bar{\tau}_{yz}\phi_{,y}) \, d\xi \, d\eta. \tag{25}$$

Equations (24) and (25) together with eqns (5e), (6a,b,c), and (14a,b,c) imply, as a system of one-dimensional constitutive relations, the two matrix equations

$$\begin{bmatrix} F \\ M_x \\ M_y \\ R \\ N \\ H \end{bmatrix} = \begin{bmatrix} A_E & S_{xE} & S_{yE} & 0 & I_{pE} & J_E \\ S_{xE} & I_{xE} & K_E & \Gamma_{xE} & I_{xpE} & J_{xE} \\ S_{yE} & K_E & I_{yE} & \Gamma_{yE} & I_{ypE} & J_{yE} \\ 0 & \Gamma_{xE} & \Gamma_{yE} & \Gamma_{\phi E} & \Gamma_{pE} & J_{\phi E} \\ I_{pE} & I_{xpE} & I_{ypE} & \Gamma_{pE} & I_{ppE} & J_{pE} \\ J_E & J_{xE} & J_{yE} & J_{\phi E} & J_{pE} & J_{\omega E} \end{bmatrix} \begin{bmatrix} \epsilon_F \\ \kappa_x \\ \kappa_y \\ \kappa_R \\ \epsilon_N \\ \gamma_H \end{bmatrix}, \tag{26}$$

and

$$\begin{bmatrix} Q_x \\ Q_y \\ T \\ S \end{bmatrix} = \begin{bmatrix} A_G & 0 & -S_{yG} & S_{yG} \\ 0 & A_G & S_{xG} & -S_{xG} \\ -S_{yG} & S_{xG} & I_{pG} & -J_G \\ S_{yG} & -S_{xG} & -J_G & J_G \end{bmatrix} \begin{bmatrix} \gamma_x \\ \gamma_y \\ \kappa_T \\ \gamma_S \end{bmatrix}, \tag{27}$$

with the elements of the constitutive matrices in (26) and (27) being given by

$$(A_E, S_{xE}, S_{yE}, I_{xE}, I_{yE}, K_E) = \iint (1, x, y, x^2, y^2, xy)E \, d\xi \, d\eta \tag{28}$$

$$(\Gamma_{xE}, \Gamma_{yE}, \Gamma_{\phi E}, \Gamma_{pE}) = \iint (x, y, \phi, x^2 + y^2)\phi E \, d\xi \, d\eta, \tag{29}$$

$$(I_{pE}, I_{xpE}, I_{ypE}, I_{ppE}) = \iint (1, x, y, x^2 + y^2)(x^2 + y^2)E \, d\xi \, d\eta, \tag{30}$$

$$(J_E, J_{xE}, J_{yE}, J_{\phi E}, J_{pE}, J_{\omega E}) = \iint [1, x, y, \phi, x^2 + y^2, (\eta\phi_{,\xi} - \xi\phi_{,\eta})] \times (\eta\phi_{,\xi} - \xi\phi_{,\eta})E \, d\xi \, d\eta, \tag{31}$$

and

$$(A_G, S_{xG}, S_{yG}, I_{pG}, J_G) = \iint (1, x, y, x^2 + y^2, \eta\phi_{,\xi} - \xi\phi_{,\eta})G \, d\xi \, d\eta, \quad (32)$$

with x and y as in eqn (4), and with the ensuing relations

$$\iint (y\phi_{,x} - x\phi_{,y})G \, d\xi \, d\eta = \iint (\eta\phi_{,\xi} - \xi\phi_{,\eta})G \, d\xi \, d\eta, \quad (33a)$$

and

$$\iint \phi_{,x}G \, d\xi \, d\eta = \iint yG \, d\xi \, d\eta, \quad \iint \phi_{,y}G \, d\xi \, d\eta = -\iint xG \, d\xi \, d\eta. \quad (33b)$$

We note that the system of differential equations involving (14a,b,c), (16) to (19), (26), and (27) reduces to our earlier results for prismatical beams [4, 5] upon stipulating that $\omega = 0$, identically, whereupon $\gamma_H = 0$ in (14b) and $\omega' H = 0$ in eqn (19), with this entailing a reduction of the 6×6 system in eqn (26) to a 5×5 system, by way of a deletion of the last row and the last column in the coefficient matrix in eqn (26).

We further note that we may, as for the case $\omega' = 0$[4], obtain a somewhat simpler theory by stipulating that transverse shear deformability is negligible, insofar as the effect of the stress measures Q_x and Q_y is concerned, by stipulating in eqn (27) that $A_G = \infty$ and

$$\gamma_x = 0, \qquad \gamma_y = 0, \quad (34)$$

with Q_x and Q_y then being reactive, and with the remainder of the system (27) reducing to

$$T = I_{pG}\kappa_T - J_G\gamma_S, \qquad S = J_G(\gamma_S - \kappa_T). \quad (35)$$

As also done in [1, 2], when $\omega' \neq 0$, we do not make the further assumption here of neglecting transverse shear deformability in relation to the magnitude of S. Instead, we retain the distinction between κ_T and γ_S as in eqn (35).

STRETCHING, TWISTING AND WARPING OF DOUBLY SYMMETRIC CROSS-SECTION BEAMS

We consider a uniform pretwisted beam of length $2L$ with a doubly symmetric cross-section, acted upon by forces F_z and moments M_z at the ends $z = \pm L$, with M_x, M_y, F_x, and F_y stipulated to vanish. Because of the assumed double symmetry we have then that M_x, M_y, Q_x, Q_y, α, β, u, and v vanish throughout, with the equilibrium equations (16)–(19) reducing to the three relations

$$F = F_z, \quad T + \theta' N = M_z, \quad R' - S - \omega' H = 0. \quad (36)$$

In these F, T, N, R, S, and H follow from eqns (26) and (27), with

$$\epsilon_F = w', \quad \kappa_T = \theta', \quad \epsilon_N = \tfrac{1}{2}(\theta')^2, \quad \kappa_R = \lambda', \quad \gamma_s = \lambda, \quad \gamma_H = \omega'\lambda \quad (37)$$

in the form

$$F = A_E w' + \tfrac{1}{2}I_{pE}(\theta')^2 + J_E\omega'\lambda, \quad (38)$$

$$N = I_{pE}w' + \tfrac{1}{2}I_{ppE}(\theta')^2 + J_{pE}\omega'\lambda, \quad (39)$$

$$H = J_E w' + \tfrac{1}{2}J_{pE}(\theta')^2 + J_{\omega E}\omega'\lambda, \quad (40)$$

$$R = \Gamma_{\phi E}\lambda', \qquad T = I_{pG}\theta' - J_G\lambda, \qquad S = J_G(\lambda - \theta'). \quad (41)$$

An inspection of eqn (36) in conjunction with eqns (38)–(41) indicates that the first two relations in eqn (36) are in effect two nonlinear equations for the determination of w' and θ' in terms of F_Z, M_z, and λ. The introduction of this result into the third relation in eqn (36), with R, S, and H, as in eqns (41) and (40), then leaves a second-order differential equation for the determination of λ. As regards the solution of this differential equation, the following two cases will be of particular interest,

$$\text{(i)}\quad \lambda(\pm L) = 0; \qquad \text{(ii)}\quad R(\pm L) = 0. \tag{42}$$

It is evident that for case (i) it will be necessary to determine λ as solution of a nonlinear second-order boundary value problem, explicitly or by numerical procedures.

As regards case (ii), one finds the simpler result that both the differential equation and the boundary conditions are satisfied upon setting $\lambda' = 0$ throughout, with the three relations in eqn (36) then becoming three simultaneous ordinary equations for w', θ', and λ, of the form

$$A_E w' + \tfrac{1}{2}I_{pE}(\theta')^2 + J_E\omega'\lambda = F_z, \tag{43}$$

$$I_{pG}\theta' - J_G\lambda + [I_{pE}w' + \tfrac{1}{2}I_{ppE}(\theta')^2 + J_{pE}\omega'\lambda]\theta' = M_z, \tag{44}$$

$$J_G(\lambda - \theta') - \omega'[J_E w' + \tfrac{1}{2}J_{pE}(\theta')^2 + J_{\omega E}\omega'\lambda] = 0. \tag{45}$$

Upon setting $\omega' = 0$, this system reduces to the corresponding result in [4]. Upon linearization, the consequences of eqns (43)–(45) are consistent with the developments in [2]. For the nonlinear case with $\omega' \neq 0$, we can use eqn (45) so as to express λ in terms of w' and θ', with eqns (43) and (44) then becoming a system of two simultaneous nonlinear equations for the determinations of w' and θ' in terms of F_z and M_z.

TORSION AND FLEXURE OF A PRETWISTED CANTILEVER BEAM

In an extension of earlier work[3], we now use the contents of eqns (14), (16) to (23), (26), and (27) for a consideration of the problem of a cantilever, fixed at $z = L$ and acted upon by forces F_{x0}, F_{y0}, and a torque M_{z0} at the end $z = 0$. While it is feasible to obtain results for this problem on the basis of the complete nonlinear system of equations as stated, we will here limit ourselves to the consideration of its linearized version. Furthermore, we assume for simplicity's sake that transverse shear deformability is negligible and that the origin of the x-, y-axis system coincides with the centroid of the cross section.

Given the above stipulations, we immediately obtain from eqns (16) to (22) the same as for the problem of the beam without pretwist

$$F = 0,\quad Q_x = F_{x0},\ Q_y = F_{y0},\quad M_x = F_{x0}z,\quad M_y = F_{y0}z,\quad T = M_{z0}, \tag{46}$$

with the equilibrium differential equation (19), and one of the stress boundary conditions in eqn (22) remaining in the form

$$R' - S - \omega' H = 0,\quad R(0) = 0. \tag{47}$$

With eqn (46) and with the assumed choice of axes and eqns (14a,b), we then have, as the linearized version of the constitutive system (26),

$$A_E w' + J_{\omega E}\omega'\lambda = 0,\quad H = J_E w' + J_{xE}\alpha' + J_{yE}\beta' + J_{\phi E}\lambda' + J_{\omega E}\omega'\lambda, \tag{48}$$

$$I_{xE}\alpha' + K_E\beta' + \Gamma_{xE}\lambda' + J_{xE}\omega'\lambda = F_{x0}z, \tag{49a}$$

$$K_E\alpha' + I_{yE}\beta' + \Gamma_{yE}\lambda' + J_{yE}\omega'\lambda = F_{y0}z, \tag{49b}$$

$$R = \Gamma_{xE}\alpha' + \Gamma_{yE}\beta' + \Gamma_{\phi E}\lambda' + J_{\phi E}\omega'\lambda. \tag{50}$$

In view of the assumption of absent transverse shear deformability, the associated system (27), with eqns (35) and (14b,c), reduces to

$$I_{pG}\theta' - J_G\lambda = M_{z0}, \qquad S = J_G(\lambda - \theta'). \tag{51}$$

For the problem as stated, the sixth-order problem (47) to (51) for the five dependent variables w, α, β, θ, λ is associated with the five displacement boundary conditions

$$w(L) = \alpha(L) = \beta(L) = \theta(L) = \lambda(L) = 0, \tag{52}$$

in addition to the one stress boundary condition in (47).

With the solution of the above, we can subsequently determine u and v from $u' + \alpha = 0$, $v' + \beta = 0$, which follow from eqns (34) and (14c), in conjunction with the conditions $u(L) = v(L) = 0$.

The problem as it stands now requires that we solve eqns (49a,b) for α' and β', as linear combinations of $F_{x0}z$, $F_{y0}z$, λ', and λ, with the substitution of these expressions into eqns (48) and (50), giving H and R as linear combinations of $F_{x0}z$, $F_{y0}z$, λ', and λ. The introduction of these into eqn (47), with $I_{pG}S = J_G(D_{tG}\lambda - M_{z0})$, where $D_{tG} = I_{pG} - J_G$ in accordance with eqn (51), altogether leaves a second-order differential equation for λ with the two boundary conditions $R(0) = \lambda(L) = 0$ and with the solution λ coming out as a linear combination of M_{z0}, F_{x0}, F_{y0}. Having determined λ, we will then have further, on the basis of eqn (51), an expression for θ' of the form

$$\theta' = M_{z0}f_{z0}(z) + F_{x0}f_{x0}(z) + F_{y0}f_{y0}(z), \tag{53}$$

with the functions f depending on cross-sectional properties, as well as on the rate of pretwist function ω'.

Given eqn (53) we can, as for the problem without pretwist[3], determine coordinates x_T, y_T of cross-sectional centers of twist, or equivalently, centers of shear x_S, y_S upon setting $M_{z0} = F_{y0}x_T - F_{y0}y_T$ for $\theta(0) = 0$, in the form $x_T x_S = \int_0^L (f_{y0}/f_{z0})\,dz$ and $y_T = x_S = -\int_0^L (f_{x0}/f_{z0})\,dz$.

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